AN ITERATIVE FILTER FOR SOLUTION OF THE INVERSE HEAT-CONDUCTION PROBLEM

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An iterative modification of the nonlinear Kalman filter is proposed for the determination of time-variable heat-transfer coefficients.

Statistical data-processing methods are used extensively in various branches of science and engineering; one such method is optimal dynamic filtering [1], which is highly amenable to computer implementation. This method makes it possible, given sparse data on the object of investigation, to obtain an optimal estimate of the state vector from the standpoint of minimizing the rms error, without requiring large space in computer memory, since the information stored in each computational cycle turns out to be quite limited.

An intriguing idea is the application of a Kalman filter for the solution of various heat-conduction problems, including inverse problems [2, 3], the investigation of which has taken on special significance lately in connection with the growing demands on engineering calculations, including calculations of temperature fields and stresses. The indicated method is particularly appropriate insofar as it exhibits acceptable accuracy even in situations where reliable data on the boundary and initial conditions are not available.

Here we propose a modification of the Kalman filter whereby, in essence, the conventional filter relations are used repeatedly in each time step. This approach to the solution of the problem makes it possible to obtain in each step the greatest possible proximity of the estimated values to the standard values, since the transition matrices $\Phi_{k+1,k}$, $F_{k+1,k}$ and $G_{k+1,k}$ in the initial equation of the investigated system

$$\mathbf{X}_{k+1} = \Phi_{k+1,k}\mathbf{X}_k + F_{k+1,k}\mathbf{U}_k + G_{k+1,k}\mathbf{W}_k \tag{1}$$

are refined with each subsequent iteration with allowance for the estimator values obtained for the state vector in the preceding iteration.

In Eq. (1) $\mathbf{X}_{k+1}$ is the extended state vector, which includes the temperature field vector $\mathbf{T}_{k+1}$ and the vector of boundary parameters α_{k+1}. Both of these vectors are unknowns under the conditions of the given problem and are to be considered.

Denoting by $\hat{\mathbf{X}}^{(j)}_{k+1\,k+1}$ the estimate obtained for the state vector after the j-th iteration in the (k + 1)-th time step, we write the following algorithm for the iterative filter (with covariance matrix $Q_k \equiv 0$):

$$\hat{\mathbf{X}}^{(j)}_{k+1|k+1} = \hat{\mathbf{X}}^{(j|j-1)}_{k+1,k+1} + K^{(j)}_{k+1}\left[\mathbf{Y}_{k+1} - H_{k+1}\hat{\mathbf{X}}^{(j|j-1)}_{k+1|k+1}\right], \tag{2}$$

$$P^{(j)}_{k+1|k} = \Phi^{(j-1)}_{k+1,k}P_{k,k}\Phi^{T(j-1)}_{k+1,k}, \tag{3}$$

$$K^{(j)}_{k+1} = P^{(j)}_{k+1,k}H^T_{k+1}\left[H_{k+1}P^{(j)}_{k+1|k}H^T_{k+1} + R_{k+1}\right]^{-1}, \tag{4}$$

where $\mathbf{Y}_{k+1}$ is the vector of measurements, H_{k+1} is the corresponding matrix, R_{k+1} is the covariance matrix of the measurement errors, and $\Phi^{(j-1)}_{k+1,k}$, $P^{(j)}_{k+1|k}$, $K^{(j)}_{k+1}$ are the transition and covariance matrices of the prediction errors and a weighting matrix, respectively, all of which are refined in the course of iteration.

The matrix $\Phi^{(j-1)}_{k+1,k}$ is constructed in accordance with Eq. (1), starting with the matrix form of the finite-difference equation (implicit scheme)

$$A\left[\mathbf{T}(\tau_{k-1})\right]\mathbf{X}(\tau_k) = C\left[\mathbf{T}(\tau_{k-1})\right]\mathbf{X}(\tau_{k-1}) + D\left[\mathbf{T}(\tau_{k-1})\right]\mathbf{U}(\tau_k) + Q(\tau_k)\mathbf{W}(\tau_k), \tag{5}$$

in which all the unknowns (temperature and heat-transfer coefficient) are included in the estimated vector **X**.

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The prediction vector $\hat{\mathbf{X}}^{(j|j-1)}_{k+1|k+1}$ from the (j − 1)-th to the j-th iteration is computed according to the expression

$$\hat{\mathbf{X}}^{(j|j-1)}_{k+1|k+1} = \Phi'^{(j-1)}_{k+1,k}\hat{\mathbf{X}}^{(j)}_{k|k} + F'^{(j-1)}_{k+1,k}\mathbf{U}'_k, \tag{6}$$

in which the vector X includes the parameter to be identified

$$\hat{\alpha}^{(j|j-1)}_{k+1|k+1} = \mathbf{f}(\tau_{k+1}). \tag{7}$$

We assume here that the iterative process in the k-th step terminates in the j-th iteration and the transition matrices $\Phi'^{(j-1)}_{k+1,k}$ and $F'^{(j-1)}_{k+1,k}$, which are also refined in the iterative process, are formed in accordance with the equation

$$A_1[\mathbf{X}(\tau_{k-1})]\,\mathbf{T}(\tau_k) = C_1[\mathbf{X}(\tau_{k-1})]\,\mathbf{T}(\tau_{k-1}) + D_1[\mathbf{X}(\tau_{k-1})]\,\mathbf{U}^1(\tau_k) + Q(\tau_k)\,\mathbf{W}(\tau_k), \tag{8}$$

in which $\mathbf{T}(\tau_k)$ is the temperature vector and the unknown estimated parameters are replaced by their estimated values.

The most complex problem is the selection of the function f_1, the form of which is unknown, along with the determination of the prediction vector $\hat{\alpha}_{k+1|k}$. One advantage of the iterative filter modification is the fact that, being able to refine the vector $\hat{\alpha}^{(j|j-1)}_{k+1|k+1}$ from iteration to iteration, we can use in place of (7) the equation

$$\hat{\alpha}^{(j|j-1)}_{k+1,k+1} = \hat{\alpha}^{(j-1)}_{k+1|k+1} \tag{9}$$

not only within the time step, but also the analogous equation

$$\hat{\alpha}_{k+1|k} = \hat{\alpha}^{(j)}_{k|k} \tag{10}$$

in transition to the next time step.

Referring to the iterative filter algorithm, we note that in finding the estimate of the state vector for the first iteration of the current step $\hat{\mathbf{X}}^{(1)}_{k+1|k+1}$ we take as the prediction of the estimate of the temperature field vector the usual prediction vector $\hat{\mathbf{T}}_{k+1|k}$ (see, e.g., [3]) and compute the covariance matrix of the refined estimate according to the equation

$$P_{k|k} = P^{(j)}_{k|k-1} - K^{(j)}_k H_k P^{(j)}_{k|k-1}. \tag{11}$$

Thus, the algorithm for the iterative Kalman filter entails the sequential application of Eqs. (11), (6), (9), (3), (4), and (2).

We now consider the selection of a criterion for stopping of the iterative process. There are several possible stopping criteria. First, the iterations can be stopped after some set number of them, but this approach is not reliable enough, because the required number of iterations can differ substantially at different times. Second, this stopping criterion can be an estimate of the norm of the difference of the vectors $\mathbf{Y}_{k+1}$ and $H_{k+1}\hat{\mathbf{X}}^{(j)}_{k+1|k+1}$, where the estimated-state vector $\hat{\mathbf{X}}_{k+1|k+1}$ is interpreted as the result of the last iteration. This criterion is written formally

$$\|\mathbf{Y}_{k-1} - H_{k-1}\hat{\mathbf{X}}^{(j)}_{k-1|k-1}\| < \varepsilon.$$

Finally, a third criterion is an estimate of the norm of the difference of the results of two successive iterations:

$$\|\hat{\mathbf{T}}^{(j)}_{k-1|k-1} - \hat{\mathbf{T}}^{(j-1)}_{k-1|k-1}\| < \varepsilon_1.$$

This criterion is the most practical and is therefore the most commonly used.

To speed convergence of the iterative process we can use a method based on the calculation of the scalar performance figure of a filter [4]

$$J^{(j)}_{k-1} = [H_{k-1}\hat{\mathbf{X}}^{(j|j-1)}_{k-1|k-1} - \mathbf{Y}_{k-1}]^T\,[H_{k-1}P^{(j)}_{k-1|k}H^T_{k-1} - R_{k-1}]^{-1}[H_{k-1}\hat{\mathbf{X}}^{(j|j-1)}_{k-1|k-1} - \mathbf{Y}_{k-1}], \tag{12}$$

which is the quadratic form of the predicted measurement errors.

Making use of the properties of a normal Gaussian distribution of random variables, which governs the deviation of a measurement from its standard value, and knowing that the value of this deviation is with

probability 0.997 not greater than 3σ (where σ characterizes the deviations of the measured parameters), we can estimate the threshold value of this quantity J, which is equal to 9.

The speedup of convergence entails selection of a special weighting factor

$$S_{k+1}^{(j)} = \frac{J_{k+1}^{(j)}}{9}, \tag{13}$$

which increases the elements of the weighting matrix, acting on the covariance matrix of the prediction errors with the corresponding replacement of expression (3) by the equation

$$P_{k+1|k}^{(j)} = S_{k+1}^{(j)} \Phi_{k+1,k}^{(j-1)} P_{k|k} \Phi_{k+1,k}^{T(j-1)}. \tag{14}$$

The application of this method makes it possible to decrease the number of iterations and, accordingly, to diminish the time of solution. An essential consideration, particularly in the solution of the inverse heat-conduction problem, is the equation of the errors of measurements distributed according to a normal Gaussian law with variance σ^2. The convergence of the filter takes place in a so-called convergence tube, whose diameter in each step is determined by the elements of the matrix $P_{k/k}$, which decrease with time (the norm of $P_{k/k}$ tends to the norm of the covariance matrix of the measurement errors R_k). Thus, the estimated values can converge, rather than to the standard, to values that differ from the standard values by the amount of the measurement error (in conversion of the measured parameters into estimated parameters).

In connection with the fact that, as tests have shown, the errors of temperature measurements can have a significant influence on the convergence of the estimated vector α we encounter the problem of smoothing of the measurements, i.e., their preliminary refinement prior to application of the filter algorithm. The following scheme is proposed as the method of smoothing of the measurements. At a given measured point a small symmetric "smoothing" time interval is chosen, and inside this interval additional measurements of the temperature at the given point are made. If, for example, n measurements are made in this interval, then the following quantity is chosen as the measurement to be used in the filter algorithm:

$$Y = \frac{y_1 + y_2 + \ldots + y_n}{n}, \tag{15}$$

where $y_1, \ldots, y_n$ are the measurements inside the given time interval.

Studies have shown that the deviation characteristic σ_s obtained after the smoothing process differs from the normal characteristic σ by a factor $\sqrt{n}$, i.e., $\sigma_s = \sigma/\sqrt{n}$.

Convergence can also be improved by partitioning the first time step into substeps, in each of which it is possible to use both the iterative and the noniterative filter modifications. This artifice has the effect of bringing the estimator values of the parameter to be identified close to the standard even in the first time steps.

As an illustration characterizing the efficiency of the proposed procedure we give the example of identification of the boundary conditions (determination of the heat-transfer coefficient α) for an infinite plate of thickness L = 0.04 m with simultaneous reconstruction of its temperature field for an unknown initial distribution.

At one boundary we adopt zero-valued Neumann boundary conditions ($\partial T/\partial n = 0$) and assume that Cauchy boundary conditions hold at the other boundary. Here the temperature of the medium is $T_m = 600°C$. In regard to the heat-transfer coefficients we identify two different functions $\alpha_1(\tau)$ and $\alpha_2(\tau)$. As the standard functions used to obtain the "measurements" we adopt

$$\alpha_1(\tau) = \frac{50 - 0.03T_m}{L}\left\{1 + 0.5\frac{1.23\cdot 10^{-5} - 1.05\cdot 10^{-8}T_m}{L^2}\tau - 0.5\exp\left[-(1.23\cdot 10^{-5} - 1.05\cdot 10^{-8}T_m)\tau/L^2\right]\right\} \left[\mathrm{W/m^2\cdot °K}\right]$$

and the function $\alpha_2(\tau)$ shown in Fig. 1. The temperature dependence of the thermal conductivity and thermal diffusivity are given by the expressions

$$\lambda = 50 - 0.03T \ [\mathrm{W/m\cdot °K}];$$

$$a = 1.23\cdot 10^{-5} - 1.05\cdot 10^{-8}T \ [\mathrm{m^2/sec}].$$

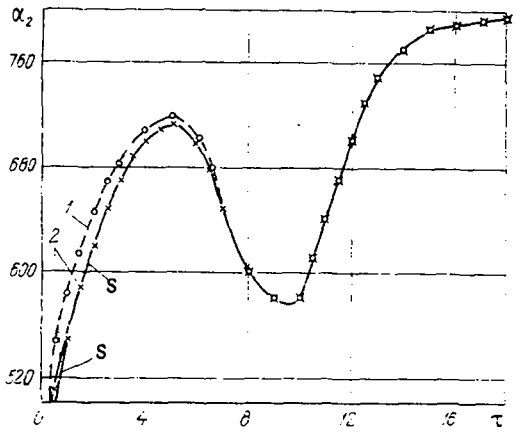

Fig. 1. Results of identification of the heat-transfer coefficient (α_2, W/m^2·deg K, τ, h; S = standard dependence).

We work with a space–time computing grid using steps h = 0.01 m (near the boundaries h = 0.005 m) and $\Delta\tau$ = 30 min. The measured parameter is the temperature at the boundary node, where Cauchy boundary conditions hold.

The estimate of the initial state vector is chosen arbitrarily. In particular, we use: $\hat{T}_{i0/0} = 50°C$; $\hat{\alpha}_{0/0} = 200$ [W/m^2·deg K].

The results of identification of the function $\alpha_2(\tau)$ are given in Fig. 1 in which the estimator curve 1 corresponds to the case where the first time step is not partitioned into substeps, and curve 2, which actually merges quickly with the standard curve (S), is obtained after partitioning of the first time step into 10 substeps ($\Delta\tau_1$ = 3 min). As for the identification of $\alpha_1(\tau)$, the estimator curve 1 with partitioning of the first step into 5 substeps ($\Delta\tau_1$ = 6 min) follows the standard curve so closely that the difference between them cannot be discerned even by large-scale plots (Fig. 2). The reconstruction of the temperature field of the plate reveals that the deviation of the estimator values from the standard values is not greater than 0.5 to 1.0% even in the first time steps.

It would be natural to expect the same effect to be attainable by means of a noniterative filter with a corresponding partition of the time steps. The solution of individual problems shows, however, that such a change in the procedure increases the time of solution and can result in a deterioration of convergence as well as the acquisition of results differing from the standard values. This fact is attributable to several considerations.

First, the number of iterations in the iterative modification differs appreciably at different times, eliciting a variable time step in the noniterative filter in different parts of the nonsteady process (an effect that complicates the solution algorithm considerably).

Second, the time of solution increases abruptly, since the covariance matrix $P_{k/k}$ in the iterative filter is determined only once for each step, whereas the application of the noniterative filter with a correspondingly smaller time step requires the determination of $P_{k/k}$ for each (shorter) step.

Third, as we know from filtering theory, the norm of the covariance matrix $P_{k/k}$ and, accordingly, the normal of the weighting matrix K_{k+1} represent monotonically decreasing functions of the number of measurements, weakening the effect of new measurements on the progress of the solution, i.e., investing the filter with a certain "sluggishness," which naturally affects its convergence.

The solutions obtained in the identification of the function $\alpha_1(\tau)$ by means of iterative (curve 1) and noniterative (curve 2) filters are compared in Fig. 2. In time steps 3-7 curve 2 practically coincides with curve 1 and the standard (S) curve $\alpha_1(\tau)$, and then in steps 12-14 it already begins to depart from it. By the 24th step the deviation is 25%.

It is important to note, however, that in the given comparison the iterative filter modification is not against the noniterative modification under the condition that the algorithm for the latter does not include measures for improving the convergence of the filter, and such measures would unquestionably improve the results obtained by the noniterative scheme (several publications touch on techniques for the improvement of convergence; see, e.g., [3]).

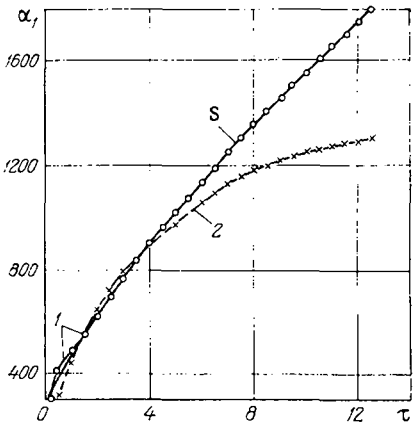

Fig. 2

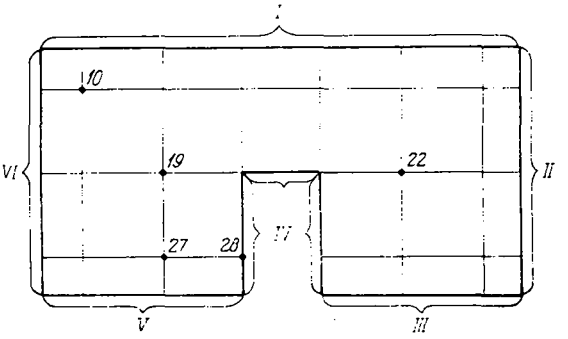

Fig. 3

Fig. 2. Comparison of solutions obtained by means of iterative and noniterative filters (α_1, W/m^2·deg K; τ, h).

Fig. 3. Measurement points in the investigated body (boundary conditions). 1) $\partial T/\partial n = 0$; II) $T = \text{const}$; III) $\partial T/\partial n = 0$; IV) $\alpha = 5000$ W/m^2·deg K, $T_0 = 373$°K; V) $\partial T/\partial n = 0$; VI) $\alpha = 11{,}400$ W/m^2·deg K, $T_m = 1073$°K.

TABLE 1. Dependence of Estimate Errors ($\hat{\alpha}_{k/k} - \alpha_S$) on Measurement Errors

σ, °K	τ_k, h							
	1	2	3	4	5	6	7	8
0,003	54	26	13	7	4	2,3	1,3	0,8
0,3	54	37	21	10,6	6	2,8	2,4	2,1
1,0	—37	25	27	8	5	—4,5	—3,6	3,3

σ, °K	τ_k, h							
	9	10	11	12	13	14	15	16
0,003	0,6	—0,3	—0,3	—0,5	—0,3	—0,5	—0,6	—0,8
0,3	0,7	—4,9	—1,3	—3,8	0,9	—6,8	4,2	0,1
1,0	—1	—5,6	—3,1	—9	4,3	—5,6	8,7	1

We have carried out a practical analysis of the stability of the solutions obtained here. Accordingly, the arbitrarily selected initial estimates of the state vector are varied over a wide range (the temperature from −50 to +100°C; the heat-transfer coefficient from 100 to 500 W/m^2·deg K; the time step from 60 to 0.3 min), along with the composition of the measurement vector. Also, we have investigated the influence of possible errors in the initial data, i.e., measurement errors, on the final optimal estimates. Whereas the influence of the initial estimates and time step is practically imperceptible (at any rate within the limits of a 200-fold variation of $\Delta\tau$ the disparity in the estimates cannot be discerned in the plotted curves), the measurement errors induce a certain scatter of the estimates (see Table 1), which, however, cannot in any sense of the word qualify as a sign of incorrectness. If we allow for the fact that the procedure provides a facility for smoothing of the measurements in any time step, the solution turns out to be practically stable.

To illustrate the invariance properties of the procedure under the selected measurement vector we consider the more typical two-dimensional problem of nonlinear nonsteady heat conduction, where simultaneously with identification of the heat-transfer boundary conditions the temperature field is reconstructed (Fig. 3). The results of the solution, which are given in Fig. 4, show that convergence of the estimates of the identified parameter α to the standard value 5000 W/m^2·deg K is observed independently of whatever composition is chosen for the measurement vector. The measurements were performed at nodes 10, 19, 22, and 28 of the grid (curve 1) and at nodes 10, 19, 22, and 27 (curve 2).

The numerical experiment indicates practical regularity on the part of the resulting solutions in the case $\alpha = \text{const}$. The stability of the solution for a time-variable heat-transfer coefficient is currently under investigation and will be the subject of a special discussion.

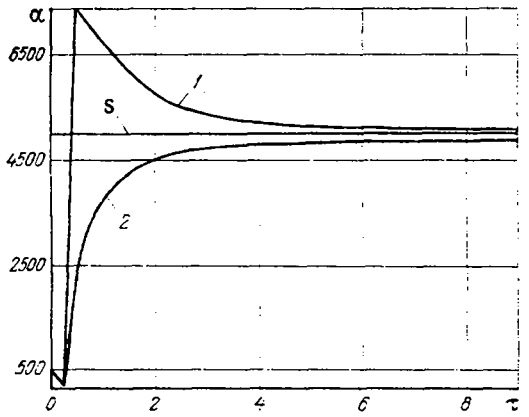

Fig. 4. Results of identification of the heat-transfer coefficient in the two-dimensional problem (α, W/m^2·deg K; τ, h).

Thus, the use of an iterative filter is adequately effective for the solution of direct and inverse heat-conduction problems, particularly in conjunction with smoothing of the measurements and partitioning of the first time steps into smaller substeps.

NOTATION

X_k, T_k, α_k	are the state vectors;
Φ_{k+1}, $F_{k+1,k}$, $G_{k+1,k}$, $\Phi'_{k+1,k}$, $F'_{k+1,k}$	are the transition matrices;
U_k, W_k	are the control and noise vectors, respectively;
$P_{k/k}$, $P_{k+1/k}$, R_{k+1}	are the covariance matrices;
K_{k+1}	is the weighting matrix;
H_{k+1}	is the matrix of measurements;
A, C, D, Q	are the matrices;
V_{k+1}	are the vector measurements;
$\|\cdot\|$	is the matrix norm;
J_k	is the scalar performance figure;
S_{k+1}	is the weighting factor;
α	is the heat-transfer coefficient;
λ	is the thermal conductivity;
a	is the thermal diffusivity;
$\Delta\tau$	is the time step;
h	is the space step of the computing grid.

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